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The regular indefinite linear-quadratic problem with linear endpoint constraints

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Abstract: This paper deals with the infinite horizon linear-quadratic problem with indefinite cost. Given a linear system, a quadratic cost functional and a subspace of the state space, we consider the problem of minimizing the cost functional over all inputs for which the state trajectory converges to that subspace. Our results generalize classical results on the zero-endpoint version of the linear-quadratic problem and more recent results on the free-endpoint version of this problem.

Keywords: Linear-quadratic problem; indefinite cost; Riccati equation; linear endpoint constraints.

1. Introduction

Consider the finite dimensional linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Given an initial point x_0 and an input function u, the state trajectory of (1.1) is denoted by $x_u(t, x_0)$. In addition to (1.1) consider the quadratic cost functional

$$J(x_0, u) = \int_0^\infty \omega(x_u(t, x_0), u(t))\,\mathrm{d}t \tag{1.2}$$

Here, $\omega(x, u)$ is a general real quadratic form on $\mathbb{R}^n \times \mathbb{R}^m$ given by, say,

$$\omega(x, u) = x^{\mathrm{T}}Qx + 2u^{\mathrm{T}}Sx + u^{\mathrm{T}}Ru, \tag{1.3}$$

with $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ symmetric and $S \in \mathbb{R}^{m\times n}$. We allow ω to be *indefinite*. It will however be a standing assumption that $R > 0$.

We shall now explain how the indefinite integral in (1.2) should be interpreted for a given u. Let $L_{2,\mathrm{loc}}(\mathbb{R}^+)$ be the space of all vector valued measurable functions u such that $\int_{t_0}^{t_1} \| u(t) \|^2\,\mathrm{d}t < \infty$ for all t_0, $t_1 \geq 0$. If $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$ then for all $T \geq 0$ the integral

$$J_T(x_0, u) := \int_0^T \omega(x_u(t, x_0), u(t))\,\mathrm{d}t$$

exists. The set of those $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$ for which $\lim_{T\to\infty} J_T(x_0, u)$ exists in $\mathbb{R}^e := \mathbb{R} \cup \{-\infty, +\infty\}$ is denoted by $U(x_0)$. For $u \in U(x_0)$ we define

$$J(x_0, u) := \lim_{T\to\infty} J_T(x_0, u) \quad (\in \mathbb{R}^e).$$

In [5], an extensive treatment was fiven of the *zero-endpoint* linear-quadratic problem associated with (1.1) and (1.2). This optimization problem is formulated as follows. For a given $x_0 \in \mathbb{R}^n$ define

$$U_0(x_0) := \left\{ u \in U(x_0) \mid \lim_{t\to\infty} x_u(t, x_0) = 0 \right\}.$$

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Find the optimal cost

$$V^{+}(x_0) := \inf\{ J(x_0, u) \mid u \in U_0(x_0)\} \tag{1.4}$$

together with all optimal inputs, i.e. all $u^* \in U_0(x_0)$ such that $J(x_0, u^*) = V^{+}(x_0)$.

Complementary to the above problem, in a recent paper [4] we resolved the *free-endpoint* linear-quadratic problem: find the optimal cost

$$V_f^{+}(x_0) := \inf\{ J(x_0, u) \mid u \in U(x_0)\} \tag{1.5}$$

together with all optimal inputs, i.e. all $u^* \in U(x_0)$ such that $J(x_0, u^*) = V_f^{+}(x_0)$.

In the present paper we shall formulate and resolve a linear-quadratic problem which has both the zero-endpoint version as well as the free-endpoint version *as special cases.* Let L be an arbitrary subspace of $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ let $d(x, L)$ be the distance from x to L. For a given initial point $x_0 \in \mathbb{R}^n$ we shall denote by $U_L(x_0)$ the subset of $U(x_0)$ consisting of those input functions u for which the state trajectory $x_u(t, x_0)$ converges to L, i.e.

$$U_L(x_0) := \left\{ u \in U(x_0) \mid \lim_{t\to\infty} d(x_u(t, x_0), L) = 0 \right\}. \tag{1.6}$$

We define the *L-endpoint* linear quadratic problem as follows: given x_0, find the optimal cost

$$V_L^{+}(x_0) := = \inf\{ J(x_0, u) \mid u \in U_L(x_0)\} \tag{1.7}$$

together with all optimal inputs, i.e. all $u^* \in U_L(x_0)$ such that $J(x_0, u^*) = V_L^{+}(x_0)$.

Clearly, the zero-endpoint problem and the free-endpoint problem can be reobtained from the latter formulation by taking $L = 0$ and $L = \mathbb{R}^n$, respectively.

2. The algebraic Riccati equation

The characterization of the optimal cost and the optimal controls for the linear quadratic problems formulated above centers around the algebraic Riccati equation (ARE):

$$A^{\mathrm{T}}K + KA + Q - (KB + S^{\mathrm{T}})R^{-1}(B^{\mathrm{T}}K + S) = 0. \tag{2.1}$$

We denote by Γ the set of all real symmetric solutions of the ARE. It was shown in [5] that if (A, B) is controllable then if $\Gamma \neq \emptyset$ it contains a unique element K^- such that $A^- := A - BR^{-1}(B^{\mathrm{T}}K^- + S)$ has all its eigenvalues in $\mathbb{C}^+ \cup \mathbb{C}^0$ and a unique element K^+ with the property that $A^+ := A - BR^{-1}(B^{\mathrm{T}}K^+ + S)$ has all its eigenvalues in $\mathbb{C}^- \cup \mathbb{C}^0$. Here we denote $\mathbb{C}^+$ ($\mathbb{C}^0$, $\mathbb{C}^-$) $:= \{s \in \mathbb{C} \mid \mathrm{Re}\ s > 0\}$ ($\mathrm{Re}\ s = 0$, $\mathrm{Re}\ s < 0$). These particular elements of Γ have the property that they are the *extremal* solutions of the ARE, in the sense that $K \in \Gamma$ implies $K^- \leq K \leq K^+$. The difference $K^+ - K^-$ is denoted by Δ. For $K \in \Gamma$ we denote $A_K := A - BR^{-1}(B^{\mathrm{T}}K + S)$.

If $M \in \mathbb{R}^{n\times n}$ then we denote by $X^+(M)$ ($X^0(M)$, $X^-(M)$) the span of all generalized eigenvectors of M corresponding to its eigenvalues in $\mathbb{C}^+$ ($\mathbb{C}^0$, $\mathbb{C}^-$). Let Ω denote the set of all A^- invariant subspaces of $X^+(A^-)$. The following well-known result states that there exists a one-to-one correspondence between Ω and Γ:

Theorem 2.1. [5,1,2]. *Let (A, B) be controllable and assume $\Gamma \neq \emptyset$. If $V \in \Omega$ then $\mathbb{R}^n = V \oplus \Delta^{-1}V^{\perp}$. There exists a bijection $\gamma : \Omega \to \Gamma$ defined by*

$$\gamma(V) := K^-P_V + K^+(I - P_V),$$

where P_V is the projector onto V along $\Delta^{-1}V^{\perp} := \{x \in \mathbb{R}^n \mid \Delta x \in V^{\perp}\}$. If $K = \gamma(V)$ then $X^+(A_K) = V$, $X^0(A_K) = \ker \Delta$ and $X^-(A_K) = X^-(A^+) \cap \Delta^{-1}V^{\perp}$. □

If $K = \gamma(V)$ then K is said to be *supported by V*.

3. Finiteness of optimal cost

For a given $x_0 \in \mathbb{R}^n$ the optimal costs $V^+(x_0)$, $V_f^+(x_0)$ and $V_L^+(x_0)$ as defined by (1.4), (1.5) and (1.7) can in principle be equal to $-\infty$ or $+\infty$. Following [5] and [4] we want to restrict ourselves to the case that the optimal costs are finite for all initial points. For the zero-endpoint problem it was shown in [5] that if (A, B) is controllable then $V^+(x_0)$ is finite for all x_0 if and only if $\Gamma \neq \emptyset$. For the free-endpoint problem it was shown in [4] that if (A, B) is controllable then $V_f^+(x_0)$ is finite for all x_0 if $\Gamma \neq \emptyset$ and $K^- \leq 0$ (see also [4], Remark 4.5.). In this section we shall establish conditions under which $V_L^+(x_0)$ is finite for all x_0.

Again let L be an arbitrary subspace of $\mathbb{R}^n$. If K is a symmetric element in $\mathbb{R}^{n\times n}$ then we shall say that K is *negative semi-definite on* L if the following conditions hold:

$$\forall x_0 \in L: \quad x_0^{\mathrm{T}} K x_0 \leq 0, \tag{3.1}$$

$$\forall x_0 \in L: \quad x_0^{\mathrm{T}} K x_0 = 0 \;\Leftrightarrow\; K x_0 = 0. \tag{3.2}$$

As an example, the matrices

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & -2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} -3 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are negative semi-definite on $\{(x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3 \mid x_3 = 0\}$. We have the following condition for finiteness of V_L^+:

Theorem 3.1. *Let (A, B) be controllable. If $\Gamma \neq \emptyset$ and K^- is negative semi-definite on L then $V_L^+(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$.*

Before giving a proof of this result, note that if $L = 0$ then the conditions (3.1) and (3.2) are fulfilled trivially. Thus we reobtain the statement that if $\Gamma \neq \emptyset$ then $V^+(x_0)$ is finite for all x_0. If $L = \mathbb{R}^n$ then (3.1) and (3.2) are equivalent to: $K^- \leq 0$. Thus we reobtain the statement that if $\Gamma \neq 0$ and $K^- \leq 0$ then $V_f^+(x_0)$ is finite for all x_0.

Our proof of Theorem 3.1 hinges on the following two lemmas:

Lemma 3.2. *Let L be a subspace of $\mathbb{R}^n$ and let H be a matrix such that $L = \ker H$. Let $K \in \mathbb{R}^{n\times n}$ be symmetric. Then K is negative semi-definite on L if and only if there exists $\lambda \in \mathbb{R}$ such that $K - \lambda H^{\mathrm{T}} H \leq 0$.*

For a proof of this we refer to the appendix.

Lemma 3.3 [1]. *Let $K \in \Gamma$. Then for all $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$ and for all $T \geq 0$ we have*

$$J_T(x_0, u) = \int_0^T \| u(t) + R^{-1}(B^{\mathrm{T}} K + S) x(t) \|_R^2 \, \mathrm{d}t + x_0^{\mathrm{T}} K^- x_0 - x^{\mathrm{T}}(T) K^- x(T),$$

where we denote $x(t) := x(t, x_0)$ and $\| v \|_R^2 := v^{\mathrm{T}} R v$. $\square$

Proof of Theorem 3.1. Let $x_0 \in \mathbb{R}^n$. Since (A, B) is controllable there is an input $u \in U_L(x_0)$ such that $J(x_0, u) < +\infty$ (in fact, one can steer from x_0 to the origin in finite time). It follows that $V_L^+(x_0) \in \mathbb{R} \cup \{-\infty\}$. Let $u \in U_L(x_0)$ be arbitrary. Let H be such that $L = \ker H$ and let $\lambda \in \mathbb{R}$ be such that $K^- - \lambda H^{\mathrm{T}} H \leq 0$. According to Lemma 3.3, for all $T \geq 0$ we have

$$J_T(x_0, u) = \int_0^T \| u(t) + R^{-1}(B^{\mathrm{T}} K^- + S) x(t) \|_R^2 \, \mathrm{d}t$$
$$+ x_0^T K^- x_0 - x^{\mathrm{T}}(T)[K^- - \lambda H^{\mathrm{T}} H] x(T) - \lambda \| H x(T) \|^2.$$

Thus, for all $T \geq 0$,

$$J_T(x_0, u) \geq x_0^{\mathrm{T}} K^- x_0 - \lambda \| Hx(T) \|^2.$$

Since $x(T)$ converges to L as $T \to \infty$, we have $Hx(T) \to 0$ $(T \to \infty)$. It follows that

$$J(x_0, u) = \lim_{T \to \infty} J_T(x_0, u) \geq x_0^{\mathrm{T}} K^- x_0.$$

The latter holds for all $u \in U_L(x_0)$ so consequently we have $V_L^+ \geq x_0^{\mathrm{T}} K^- x_0 > -\infty$. □

4. Main result

In this section we shall formulate our main result, a complete solution to the L-endpoint linear quadratic problem as formulated in Section 1. The optimal cost $V_L^+(x_0)$ will turn out to be given by a particular solution of the ARE. We shall establish necessary and sufficient conditions for the existence of optimal inputs for all initial conditions and these optimal inputs will be given in the form of a state feedback control law. In the following, if $V \subset \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ then $\langle V \mid M \rangle$ denotes the largest M-invariant subspace in V.

Again let L be an arbitrary subspace of $\mathbb{R}^n$. A key role in our treatment of the L-endpoint problem is played by the subspace

$$N(L) := \langle L \cap \ker K^- \mid A^- \rangle \cap X^+(A^-). \tag{4.1}$$

Note that $N(L)$ is an A^--invariant subspace of $X^+(A^-)$ so an element of Ω. If H is a matrix such that $L = \ker H$, then $L \cap \ker K^- = \ker[{}^{K^-}_{H}]$. Hence, $N(L)$ is the undetectable subspace (relative to $\mathbb{C}^- \cup \mathbb{C}^0$) of the system $([{}^{K^-}_{H}], A^-)$ (see [3]).

It turns out that the optimal cost $V_L^+(x_0)$ is given by the solution of the ARE supported by $N(L)$. This particular solution is denoted by

$$K_L^+ := \gamma(N(L)). \tag{4.2}$$

The following theorem is the main result of this paper:

Theorem 4.1. *Let (A, B) be controllable. Assume that $\Gamma \neq \emptyset$ and that K^- is negative semi-definite on L. Then we have*

(*i*) $V_L^+(x_0)$ *is finite for all* $x_0 \in \mathbb{R}^n$.

(*ii*) *For all* $x_0 \in \mathbb{R}^n$ $V_L^+(x_0) = x_0^{\mathrm{T}} K_L^+ x_0$.

(*iii*) *For all* $x_0 \in \mathbb{R}^n$ *there exists an optimal input* u^* *if and only if* $\ker \Delta \subset L \cap \ker K^-$.

(*iv*) *If* $\ker \Delta \subset L \cap \ker K^-$ *then for each* $x_0 \in \mathbb{R}^n$ *there exists exactly one optimal input* u^* *and, moreover, this input is given by the feedback control law* $u = -R^{-1}(B^{\mathrm{T}} K_L^+ + S)x$.

Observe that [5, Th. 7] can be reobtained from this as a special case. Indeed, if $L = 0$ then $N(L) = 0$ so $K_L^+ = K^+$ (it can be seen from Theorem 2.1 that $\gamma(0) = K^+$). The condition $\ker \Delta \subset L \cap \ker K^-$ in this case reduces to $\ker \Delta \subset 0$ or, equivalently, $\Delta > 0$. Also, [4, Th. 5.1] can be reobtained as a special case: if $L = \mathbb{R}^n$ then $N(L) = N$ and $K_L^+ = K_f^+$ (see [4,5.1 and 5.3]).

In the next section we shall give a proof of Theorem 4.1.

5. Proof of the main result

In the proof of Theorem 4.1 that we give, we shall use two lemmas that were proven in [4]. For completeness, these lemmas are reformulated in the appendix. Let K_L^+ be given by (4.2) and denote

$$A_L^+ := A - BR^{-1}(B^{\mathrm{T}} K_L^+ + S). \tag{5.1}$$

According to Theorem 2.1 we have

$$X^+(A_L^+) = N(L), \qquad X^0(A_L^+) = \ker \Delta, \qquad X^-(A_L^+) = X^-(A^+) \cap \Delta^{-1} N(L)^\perp .$$

Define $X_1 := X^+(A_L^+)$, $X_2 := X^0(A_L^+)$, $X_3 := X^-(A_L^+)$. Then we have $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3$. It is easily verified that in this decomposition A^- has the form

$$A^- = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \tag{5.2}$$

for given A_{ij}. Since $A_L^+ \mid X_1 \oplus X_2 = A^- \mid X_1 \oplus X_2$ we have

$$A_L^+ = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A'_{33} \end{bmatrix} \tag{5.3}$$

for some A_{33}. Note that $\sigma(A_{11}) \subset \mathbb{C}^+$, $\sigma(A_{22}) \subset \mathbb{C}^0$ and $\sigma(A'_{33}) \subset \mathbb{C}^-$. Since $X_1 \subset \ker K^-$, K^- has the form

$$K^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22}^- & K_{23}^- \\ 0 & K_{23}^{-T} & K_{33}^- \end{bmatrix} \tag{5.4}$$

for given K_{ij}^-. Using the facts $X_2 \oplus X_3 = \Delta^{-1} X_1^\perp$ and $X_2 = \ker \Delta$, we find that

$$\Delta = \begin{bmatrix} \Delta_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{33} \end{bmatrix},$$

for given $\Delta_{11} > 0$ and $\Delta_{33} > 0$. By applying Theorem 2.1 we then find

$$K_L^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22}^- & K_{23}^- \\ 0 & K_{23}^{-T} & K_{33}^- + \Delta_{33} \end{bmatrix}. \tag{5.5}$$

We first prove a lemma stating that K_L^+ yields a lower bound for the optimal cost $V_L^+(x_0)$:

Lemma 5.1. *Assume that (A, B) is controllable, $\Gamma \neq \emptyset$ and K^- is negative semi-definite on L. Then for all $x_0 \in \mathbb{R}^n$ and $u \in U_L(x_0)$ we have*

$$J(x_0, u) \geq x_0^{\mathrm{T}} K_L^+ x_0 + \int_0^\infty \| u(t) + R^{-1}(B^{\mathrm{T}} K_L^+ + S) x(t) \|_R^2 \, \mathrm{d}t. \tag{5.6}$$

Here, we have denoted $x(t) := x_u(t, x_0)$.

Proof. Let H be a matrix such that $L = \ker H$. Let $\lambda \in \mathbb{R}$ be such that $K^- - \lambda H^{\mathrm{T}} H \leq 0$ (see Lemma 3.2). Take an arbitrary $u \in U_L(x_0)$. It follows from Theorem 3.1 that $J(x_0, u)$ is either finite or equal to $+\infty$. If it is equal to $+\infty$ then (5.6) trivially holds. Assume therefore that $J(x_0, u)$ is finite. Applying Lemma 3.3 with $K = K^-$ yields that for all $T \geq 0$,

$$\begin{aligned} &\int_0^T \| u(t) + R^{-1}(B^{\mathrm{T}} K^- + S) x(t) \|_R^2 \, \mathrm{d}t \\ &\quad = J_T(x_0, u) - x_0^{\mathrm{T}} K^- x_0 + x^{\mathrm{T}}(T)[K^- - \lambda H^{\mathrm{T}} H] x(T) + \lambda \| Hx(T) \|^2 \\ &\quad \leq J_T(x_0, u) - x_0^{\mathrm{T}} K^- x_0 + \lambda \| Hx(T) \|^2. \end{aligned} \tag{5.7}$$

Define $v(t) := u(t) + R^{-1}(B^{\mathrm{T}}K^{-} + S)x(t)$. Since $\lim_{T\to\infty} J_T(x_0, u)$ is finite and $Hx(T) \to 0$ $(T \to \infty)$ we find that $\int_0^\infty \| v(t) \|_R^2 \, \mathrm{d}t < \infty$ so $v \in L_2(\mathbb{R}^+)$. Here $L_2(\mathbb{R}^+)$ denotes the space of all vector valued measurable functions on $\mathbb{R}^+$ such that $\int_0^\infty \| v(t) \|^2 \, \mathrm{d}t < \infty$. Again using (5.7) this implies that $\lim_{T\to\infty} x^{\mathrm{T}}(T)[K^{-} - \lambda H^{\mathrm{T}}H\}x(T)$ exists and is finite. Thus $\lim_{T\to\infty} x^{\mathrm{T}}(T)K^{-}x(T)$ exists and is finite. Also, since $K^{-} - \lambda H^{\mathrm{T}}H$ is semi-definite, $(K^{-} - \lambda H^{\mathrm{T}}H)x(T)$ and hence $K^{-}x(T)$ are bounded functions of T. Denote

$$y := \begin{bmatrix} K^{-} \\ H \end{bmatrix} x.$$

Then $y \in L_\infty(\mathbb{R}^+)$, the space of all bounded, vector valued, measurable functions on $\mathbb{R}^+$. Since $\dot{x} = Ax + Bu$, we have that x, v and y are related by

$$\dot{x} = A^{-}x + Bv, \qquad y = \begin{bmatrix} K^{-} \\ H \end{bmatrix} x.$$

Now, let $\mathbb{R}^n$ be decomposed into $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3$ as introduced above. Since $X_1 \subset \ker \left[\begin{smallmatrix} K^{-} \\ H \end{smallmatrix}\right]$ we have

$$\begin{bmatrix} K^{-} \\ H \end{bmatrix} = (0,\ D_2,\ D_3)$$

for given D_2 and D_3. Write $B = (B_1^{\mathrm{T}}, B_2^{\mathrm{T}}, B_3^{\mathrm{T}})^{\mathrm{T}}$ and $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, x_3^{\mathrm{T}})^{\mathrm{T}}$. Since X_1 is the undetectable subspace (relative to $\mathbb{C}^{-} \cup \mathbb{C}^{0}$) of the system $(\left[\begin{smallmatrix} K^{-} \\ H \end{smallmatrix}\right], A^{-})$, it is easily verified that the pair

$$\left((D_2,\ D_3), \begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} \right)$$

is $(\mathbb{C}^{-} \cup \mathbb{C}^{0})$-detectable. Since $\sigma(A^{-}) \subset \mathbb{C}^{+} \cup \mathbb{C}^{0}$ and $X_2 = X^{0}(A^{-})$, we have $\sigma(A_{22}) \subset \mathbb{C}^{0}$ and

$$\sigma \begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix} \subset \mathbb{C}^{+}.$$

Hence $\sigma(A_{33}) \subset \mathbb{C}^{+}$. Also, we have

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} v, \qquad y = (D_2,\ D_3) \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}.$$

Since $v \in L_2(\mathbb{R}^+)$ and $y \in L_\infty(\mathbb{R}^+)$, by Lemma A.1 (applied with $\mathbb{C}_{\mathrm{g}} = \mathbb{C}^{-} \cup \mathbb{C}^{0}$) we have $x_3 \in L_\infty(\mathbb{R}^+)$.

By again applying Lemma 3.3, this time with $K = K_L^{+}$, we find that for all $T \geq 0$,

$$J_T(x_0, u) = \int_0^T \| u(t) + R^{-1}(B^{\mathrm{T}}K_L^{+} + S)x(t) \|_R^2 \, \mathrm{d}t + x_0^{\mathrm{T}}K_L^{+}x_0 - x^{\mathrm{T}}(T)K_L^{+}x(T). \tag{5.8}$$

Denote $w(t) := u(t) + R^{-1}(B^{\mathrm{T}}K_L^{+} + S)x(t)$. By combining (5.4) and (5.5) we obtain that for all $T \geq 0$,

$$J_T(x_0, u) = \int_0^T \| w(t) \|_R^2 \, \mathrm{d}t + x_0^{\mathrm{T}}K_L^{+}x_0 - x_3^{\mathrm{T}}(T)\ \Delta_{33}x_3(T) - x^{\mathrm{T}}(T)K^{-}x(T). \tag{5.9}$$

Recall that $\lim_{T\to\infty} J_T(x_0, u)$ was assumed to be finite. Since $x_3(t)$ and $x^{\mathrm{T}}(T)K^{-}x(T)$ are bounded functions of T, (5.9) implies that $w \in L_2(\mathbb{R}^+)$. Again consider (5.9). Since now $J_T(x_0,u)$, $\int_0^T \| w(t) \|_R^2 \, \mathrm{d}t$ and $x^{\mathrm{T}}(T)K^{-}x(T)$ *converge* for $T \to \infty$, it follows that $\lim_{T\to\infty} x_3^{\mathrm{T}}(T)\ \Delta_{33}x_3(T)$ exists and is finite. Using the fact $\Delta_{33} > 0$ this implies that that $\lim_{T\to\infty} \| x_3(T) \|$ exists. Since $\dot{x} = Ax + Bu$, the variables x and w are related by $\dot{x} = A_L^{+}x + Bw$. Hence, by (5.3), $\dot{x}_3 = A'_{33}x_3 + B_3w$. Recall that

$w \in L_2(\mathbb{R}^+)$ and that $\sigma(A'_{33}) \subset \mathbb{C}^-$. Thus $x_3 \in L_2(\mathbb{R}^+)$. Combining this with the fact that $\| x_3(T) \|$ converges as $T \to \infty$, we find that $\lim_{T\to\infty} x_3(T) = 0$. By letting $T \to \infty$ in

$$\begin{aligned} J_T(x_0, u) &= \int_0^T \| w(t) \|_R^2 \, dt + x_0^T K_L^+ x_0 - x_3^T(T)\, \Delta_{33} x_3(T) \\ &\quad - x^T(T)[K^- - \lambda H^T H] x(T) + \lambda \| Hx(T) \|^2 \\ &\geq \int_0^T \| w(t) \|_R^2 \, dt + x_0^T K_L^+ x_0 - x_3^T(T)\, \Delta_{33} x_3(T) + \lambda \| Hx(T) \|^2 \end{aligned}$$

the desired result follows. □

Our following lemma states that by appropriate choice of $u \in U_L(x_0)$ the difference between $x_0^T K_L^+ x_0$ and the cost $J(x_0, u)$ can be made arbitrarily small:

Lemma 5.2. *Assume that (A, B) is controllable and that $\Gamma \neq \emptyset$. Then for all $x_0 \in \mathbb{R}^n$ and for all $\varepsilon > 0$ there exists $u \in U_L(x_0)$ such that $J(x_0, u) \leq x_0^T K_L^+ x_0 + \varepsilon$.*

Proof. Let H be such that $\ker H = L$. Let $\mathbb{R}^n$ be decomposed into $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3$ as above. Then we have $H = (0, H_2, H_3)$ for given H_2 and H_3. From Lemma 3.3 we have that for all $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$,

$$J_T(x_0, u) = \int_0^T \| w(t) \|_R^2 \, dt + x_0^T K_L^+ x_0 - \left(x_2^T(T), x_3^T(T) \right) \begin{bmatrix} K_{22}^- & K_{23}^- \\ K_{23}^{-T} & K_{33}^- + \Delta_{33} \end{bmatrix} \begin{bmatrix} x_2(T) \\ x_3(T) \end{bmatrix}. \tag{5.10}$$

Here, $w := u + R^{-1}(B^T K_L^+ + S)x$. Since $\dot{x} = Ax + Bu$, x and w are related by $\dot{x} = A_L^+ x + Bw$ and hence, by (5.3),

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{22} & 0 \\ 0 & A'_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_2 \\ B_3 \end{bmatrix} w. \tag{5.11}$$

Recall that $\sigma(A_{22}) \subset \mathbb{C}^0$ and $\sigma(A'_{33}) \subset \mathbb{C}^-$. Also, (5.11) is controllable. From Lemma A.2 it then follows that there exists $w \in L_2(\mathbb{R}^+)$ such that $\int_0^\infty \| w(t) \|_R^2 \, dt < \varepsilon$, $x_2(T) \to 0$ and $x_3(T) \to 0$ $(T \to \infty)$. Define

$$u := -R^{-1}\left(B^T K_L^+ + S \right) x + w.$$

Then $Hx(T) = H_2 x_2(T) + H_3 x_3(T) \to 0$ $(T \to \infty)$ and

$$J(x_0, u) = \int_0^\infty \| w(t) \|_R^2 \, dt + x_0^T K_L^+ x_0 \leq \varepsilon + x_0^T K_L^+ x_0. \qquad \square$$

We are now in a position to give a proof of Theorem 4.1.

Proof of Theorem 4.1. (i) This was already proven in Theorem 3.1.

(ii) Lemma 5.1 yields $J(x_0, u) \geq x_0^T K_L^+ x_0$ for all $u \in U_L(x_0)$. Combining this with Lemma 5.2 we obtain that $V_L^+(x_0) = x_0^T K_L^+ x_0$ for all x_0.

(iii) Let H be such that $L = \ker H$. Recall that with respect to the decomposition $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3$ we have $[\begin{smallmatrix} K^- \\ H \end{smallmatrix}] = (0, D_2, D_3)$. Let $\lambda \in \mathbb{R}$ be such that $K^- - \lambda H^T H \leq 0$.

$(\Rightarrow)$ Let x_0 be arbitrary and u^* be the corresponding optimal control, $u^* \in U_L(x_0)$. Let x^* be the corresponding optimal trajectory. By Lemma 5.1,

$$x_0^T K_L^+ x_0 = J(x_0, u^*) \geq x_0^T K_L^+ x_0 + \int_0^\infty \| u^*(t) + R^{-1}\left(B^T K_L^+ + S \right) x^*(t) \|_R^2 \, dt.$$

Hence u^* must be given by the feedback control law $u^* = -R^{-1}(B^T K_L^+ + S)x^*$ and therefore x^*

satisfies $\dot{x}^* = A_L^+ x^*$. In terms of our decomposition of $\mathbb{R}^n$ this yields $\dot{x}_3^* = A'_{33} x_3^*$. Consequently, $x_3^*(t) \to 0$ $(t \to \infty)$. According to (5.9),

$$J_T(x_0, u^*) = x_0^T K_L^+ x_0 - x_3^{*'}(T)\, \Delta_{33} x_3^*(T) - x^{*'}(T) K^- x^*(T).$$

Since $J_T(x_0, u^*) \to x_0^T K_L^+ x_0$ $(T \to \infty)$ we find that $x^{*'}(T) K^- x^*(T) \to 0$ $(T \to \infty)$. Since also $Hx^*(T) \to 0$, we find that

$$x^{*'}(T)[K^- - \lambda H^T H] x^*(T) \to 0 \quad (T \to \infty).$$

The latter implies that $(K^- - \lambda H^T H) x^*(T) \to 0$ whence $K^- x^*(T) \to 0$. From this it follows that $D_2 x_2^*(T) + D_3 x_3^*(T) \to 0$ so $D_2 x_2^*(T) \to 0$. Equivalently,

$$D_2\, e^{A_{22} T} x_2(0) \to 0 \quad (T \to \infty).$$

Since $x_2(0)$ is arbitrary, we find that $D_2\, e^{A_{22}T} \to 0$ so $D_2(Is - A_{22})^{-1}$ has all its poles in $\mathbb{C}^-$. However, $\sigma(A_{22}) \subset \mathbb{C}^0$ so it also has all its poles in $\mathbb{C}^0$. It follows that, in fact, $D_2(Is - A_{22})^{-1} = 0$ whence $D_2 = 0$. We conclude that

$$\ker \Delta = X_2 \subset \ker \begin{bmatrix} K^- \\ H \end{bmatrix} = L \cap \ker K^-.$$

$(\Leftarrow)$ Conversely, assume $\ker \Delta \subset L \cap \ker K^-$. Then we have $K_{22}^- = 0$, $K_{23}^- = 0$ (see (5.4)) and $D_2 = 0$. Define $u := -R^{-1}(B^T K_L^+ + S)x$. We claim that this feedback law yields an optimal u. Indeed, by (5.10),

$$J_T(x_0, u) = x_0^T K_L^+ x_0 - x_3^T(T)(K_{33}^- + \Delta_{33}) x_3(T).$$

Moreover, $\dot{x}_3 = A'_{33} x_3$. Thus $x_3(T) \to 0$ $(T \to \infty)$ whence $J(x_0, u) = x_0^T K_L^+ x_0$. Also, $[\begin{smallmatrix} K^- \\ H \end{smallmatrix}] x(T) = D_3 x_3(T) \to 0$ so , in particular, $Hx(T) \to 0$ $(T \to \infty)$.

(iv) The fact that $u^* = -R^{-1}(B^T K_L^+ + S)x^*$ is unique was already proven in (iii). This completes the proof. $\square$

6. Appendix

In this appendix we shall first give a proof of Lemma 3.2. Next, we shall formulate two lemmas that are used in Section 5.

Proof of Lemma 3.2. $(\Rightarrow)$ Let $x_1, \ldots, x_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $x_1, \ldots, x_r$ is a basis of $L \cap \ker K$ and $x_1, \ldots, x_s$ is a basis of L $(0 \le r \le s \le n)$. With respect to the decomposition of $\mathbb{R}^n$ corresponding to this choice of basis, K and H have matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{23}^T & K_{33} \end{bmatrix} \quad \text{and} \quad (0, 0, H_3),$$

respectively. Note that H_3 is injective. Since K is negative semi-definite on L we have $K_{22} \le 0$. Also, $x_2^T K_{22} x_2 = 0$ implies $K_{23}^T x_2 = 0$. We claim that, in fact, $K_{22} < 0$. Indeed, $x_2^T K_{22} x_2 = 0$ implies $K_{22} x_2 = 0$ and $K_{23}^T x_2 = 0$ so $(0, x_2, 0)^T \in \ker K \cap L$. Hence $x_2 = 0$. Now, with respect to the given decomposition, $K - \lambda H^T H$ has the matrix

$$M(\lambda) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{23}^T & K_{33} - \lambda H_3^T H_3 \end{bmatrix}.$$

Clearly, $K-\lambda H^{\mathrm{T}}H \leq 0$ if and only if $M(\lambda) \leq 0$. Since $H_3^{\mathrm{T}}H_3$ is regular, there exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \geq \lambda_0$ we have $K_{33}-\lambda H_3^{\mathrm{T}}H_3 < 0$. Thus, for $\lambda \geq \lambda_0$ we have: $M(\lambda) \leq 0$ if and only if the Schur complement $S(\lambda) := K_{22}-K_{23}[K_{33}-\lambda H_3^{\mathrm{T}}K_3]^{-1}K_{23}^T \leq 0$. We will show that indeed there exists $\lambda \geq \lambda_0$ such that $S(\lambda) \leq 0$. Let $\mu_{\max}(\lambda)$ be the largest eigenvaue of $S(\lambda)$. Let $v(\lambda)$ be a corresponding eigenvector with $\| v(\lambda) \| = 1$. We have

$$\mu_{\max}(\lambda) = v(\lambda)^{\mathrm{T}}S(\lambda)v(\lambda) = v(\lambda)^{\mathrm{T}}K_{22}v(\lambda) - w(\lambda)^{\mathrm{T}}\left[K_{33}-\lambda H_3^{\mathrm{T}}H_3\right]^{-1}w(\lambda),$$

where $w(\lambda) := K_{23}^{\mathrm{T}}v(\lambda)$. Note that $\| w(\lambda) \| \leq c$, where $c \in \mathbb{R}$ is independent of λ. Let $\rho_{\max} < 0$ be the largest eigenvalue of K_{22}. Then we find

$$\mu_{\max}(\lambda) \leq \rho_{\max} - w(\lambda)^{\mathrm{T}}\left[K_{33}-\lambda H_3^{\mathrm{T}}H_3\right]^{-1}w(\lambda).$$

We contend that $w(\lambda)^{\mathrm{T}}[K_{33}-\lambda H_3^{\mathrm{T}}H_3]^{-1}w(\lambda) \to 0$ as $\lambda \to \infty$. Indeed, let $\tau_{\min}(\lambda)$ and $\tau_{\max}(\lambda)$ be the smallest and largest eigenvalue of $(K_{33}-\lambda H_3^{\mathrm{T}}H_3)^{-1}$, respectively. Then

$$c^2\tau_{\min}(\lambda) \leq w(\lambda)^{\mathrm{T}}\left[K_{33}-\lambda H_3^{\mathrm{T}}H_3\right]^{-1}w(\lambda) \leq c^2\tau_{\max}(\lambda).$$

Also, by the fact that $H_3^{\mathrm{T}}H_3$ is regular, $\tau_{\min}(\lambda)$ and $\tau_{\max}(\lambda)$ converge to 0 as $\lambda \to \infty$.

($\Leftarrow$) Assume $K-\lambda H^{\mathrm{T}}H \leq 0$. Let $x \in L = \ker H$. Then $x^{\mathrm{T}}Kx = x^{\mathrm{T}}(K-\lambda H^{\mathrm{T}}H)x \leq 0$. If $x^{\mathrm{T}}Kx = 0$ then $(K-\lambda H^{\mathrm{T}}H)x = 0$. Hence $Kx = 0$. This completes the proof of Lemma 3.2. □

Lemma A.1. *Consider the system $\dot{x} = Ax + v$, $y = Cx$. Let $\mathbb{C}_g$ be a symmetric subset of $\mathbb{C}$. Assume that (C, A) is detectable (relative to $\mathbb{C}_g$). Let the state space $\mathbb{R}^n$ be decomposed into $\mathbb{R}^n = X_1 \oplus X_2$, where X_1 is A-invariant. In this decomposition, let $x = (x_1, x_2)^{\mathrm{T}}$. Assume that $\sigma(A \mid X_1) \subset \mathbb{C}_g$ and $\sigma(A \mid \mathbb{R}^n/X_1) \subset \mathbb{C}_b$. Then for every initial condition x_0 we have: if $v \in L_2(\mathbb{R}^+)$ and $y \in L_\infty(\mathbb{R}^+)$ then $x_2 \in L_\infty(\mathbb{R}^+)$.*

Proof. See [4, Lemma 5.3]. □

Lemma A.2. *Consider the system $\dot{x} = Ax + Bu$, $x(0) = x_0$. Assume that (A, B) is controllable and $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$. Then for all $\varepsilon > 0$ there exists a control $u \in L_2(\mathbb{R}^+)$ such that $\int_0^\infty \| u(t) \|^2 \, \mathrm{d}t < \varepsilon$ and $x_u(t, x_0) \to 0$ $(t \to \infty)$.*

Proof. See [4, Lemma 5.4]. □

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